Unit 5

Systems of Linear Equations and Inequalities

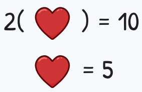
Have you ever been in a situation where you had to accomplish two goals at once? Systems of equations and inequalities are helpful for finding a set of values that meets both constraints at the same time. In this unit, you'll explore how to solve systems of equations and inequalities using different strategies. You will also analyze the structure of the equations in a system to strategize about which solving method you choose.

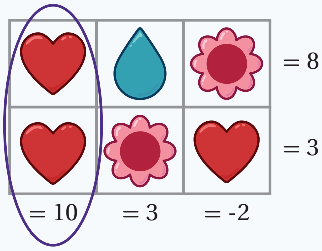
Essential Questions

- How can you solve systems of equations and inequalities symbolically and graphically?
- How can you use the structures of the equations, available tools, and knowledge of your personal mathematical preferences to select a solving method strategically?
- How can constraints be represented using systems of equations or inequalities?

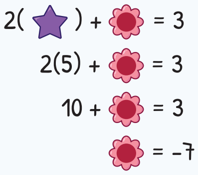
There are many different ways to solve problems and puzzles using math. Let's look at a few strategies for determining the values of shapes in a puzzle.

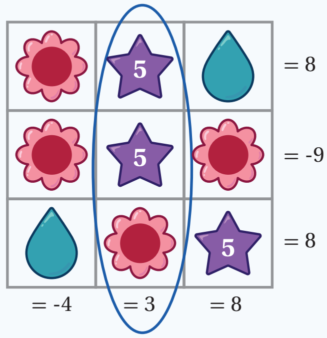
Strategy 1: Look for single-shape rows/columns
If we know the value of two hearts, we can determine the value of one heart and substitute that value in the other parts of the puzzle.



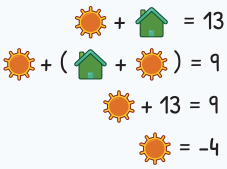


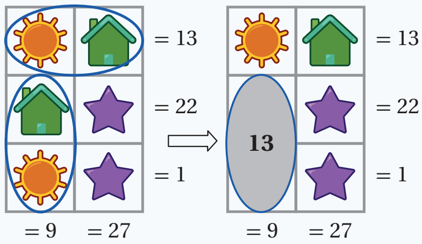
Strategy 2: Substitute known shape values to solve for missing values
If we know the value of one star, then we can substitute that value in other parts of the puzzle.





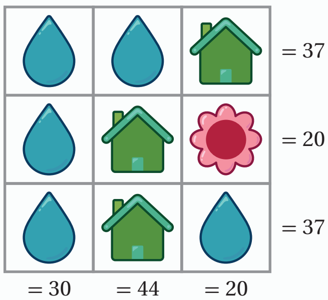
Strategy 3: Look for repeating shape patterns
If we know the value of a shape combination and we see it repeat in the puzzle, then we can substitute the value for the shape combination.





Try This

Here is a shape puzzle. The sum of each row and column is shown.



a Determine the solution for this puzzle.

Shape	Value
House	
Flower	
Drop	

b Describe the strategy you used.

Summary | Lesson 2

A **system of equations** is two or more equations that represent the same constraints using the same variables. The **solution to a system of equations** is the (x, y) ordered pair(s) that makes every equation in the system true. There are many solutions to a linear equation in two variables, but there might be only one (or no) solution to a system of linear equations in two variables.

There are many strategies for determining the ordered pair that makes both equations in a system true. One strategy is called **elimination**, where you add or subtract the equations to produce a new equation with one variable. Let's look at some examples.

If the equations in the system share the same *coefficient* with opposite signs on the same variable, you can eliminate a variable by adding. You can solve this system by adding to eliminate the y-variable.

$$\begin{array}{r} -2x + y = 9 \\ +(8x - y = 3) \\ \hline 6x + 0 = 12 \\ x = 2 \end{array}$$

$$\begin{array}{r} -2(2) + y = 9 \\ y = 13 \end{array}$$

If the equations in the system share the same *coefficient* with the same signs on the same variable, you can eliminate a variable by subtracting. You can solve this system by subtracting to eliminate the y-variable.

$$\begin{array}{r} x + 2y = 30 \\ -(x + y = 23) \\ \hline y = 7 \end{array}$$

$$\begin{array}{r} x + (7) = 23 \\ x = 16 \end{array}$$

Try This

Here is a system of equations:

$$p + 3q = 14$$

$$p + 2q = 10$$

a Circle the action(s) that can be used to eliminate a variable in this system.

Addition Subtraction Both Neither

b Determine the solution to this system of equations.

Summary | Lesson 3

It can be helpful to write *equivalent equations* when using elimination to solve systems of equations. You can create equivalent equations by multiplying each term of the first or second equation by a number. Your goal is to end up with a system of equations where one variable has the same or *opposite* coefficients so you can add or subtract them to eliminate a variable.

Here is a system of equations:

$$9x - 4y = 2$$
$$3x + y = 10$$

You can multiply the second equation by -3 to eliminate the x-variables.

$$\begin{array}{r} 9x - 4y = 2 \\ \text{-}3\,(3x + y = 10) \end{array}$$

$$\begin{array}{rr} & 9x - 4y = 2 \\ + & \text{-}9x - 3y = \text{-}30 \\ \hline & 0 - 7y = \text{-}28 \\ & \boxed{y = 4} \end{array}$$

$$\begin{array}{r} 3x + (4) = 10 \\ 3x = 6 \\ \boxed{x = 2} \end{array}$$

Or you can multiply the second equation by 4 to eliminate the y-variables.

$$\begin{array}{r} 9x - 4y = 2 \\ 4\,(3x + y = 10) \end{array}$$

$$\begin{array}{rr} & 9x - 4y = 2 \\ + & 12x + 4y = 40 \\ \hline & 21x + 0 = 42 \\ & \boxed{x = 2} \end{array}$$

$$\begin{array}{r} 9(2) - 4y = 2 \\ 18 - 4y = 2 \\ \text{-}4y = \text{-}16 \\ \boxed{y = 4} \end{array}$$

Try This

Solve this system of equations in two different ways.

System of Equations	Strategy 1	Strategy 2
$4x + 3y = 3$ $8x + y = 1$		

Summary | Lesson 4

One strategy you can use to solve a system of equations is **substitution**, where you replace a variable with an equivalent expression. Substitution is a useful strategy when one variable is already isolated in an equation.

Here are two examples of systems of equations where substitution may be a useful strategy.

In this system, both y-variables are already isolated. We can substitute the expression $-4x + 6$ in for y in the second equation.

$$y = -4x + 6$$
$$y = 3x - 15$$

$$y = -4x + 6$$

$$\begin{aligned} y &= 3x - 15 \\ -4x + 6 &= 3x - 15 \\ -7x &= -21 \\ \boxed{x} & \boxed{= 3} \\ y &= 3(3) - 15 \\ \boxed{y} & \boxed{= -6} \end{aligned}$$

In this system of equations, y is already isolated, so we can substitute the expression $2x - 5$ in for y in the first equation.

$$-3x - 2y = 3$$
$$y = 2x - 5$$

$$y = 2x - 5$$

$$\begin{aligned} -3x - 2y &= 3 \\ -3x - 2(2x - 5) &= 3 \\ -3x - 4x + 10 &= 3 \\ -7x + 10 &= 3 \\ -7x &= -7 \\ \boxed{x} & \boxed{= 1} \\ y &= 2(1) - 5 \\ \boxed{y} & \boxed{= -3} \end{aligned}$$

Try This

a Determine the solution to this system of equations.

$b = 3a - 8$

$2a + b = 2$

b How is solving systems of equations by substitution like solving by elimination? How is it different?

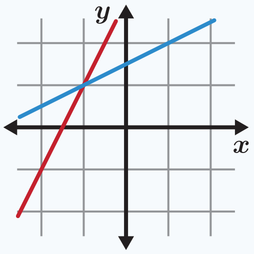
Summary | Lesson 5

You can solve systems of equations using strategies like elimination, substitution, or graphing.

On a coordinate plane, you can see the solution of a system of equations at the point(s) where the two lines intersect. A system of linear equations can have:

One Solution

The lines intersect at (-2, 2).

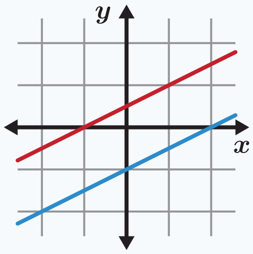


$$y = 2x + 6$$
$$y = \frac{1}{2}x + 3$$

The equations have different *slopes* and y*-intercepts*.

No Solutions

The lines are *parallel*.

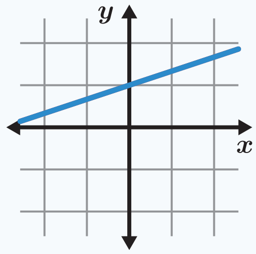


$$y = \frac{1}{2}x + 1$$
$$y = \frac{1}{2}x - 2$$

The equations have the same slope and different y-intercepts.

Infinitely Many Solutions

The lines are the same.



$$y = \frac{1}{3}x + 2$$
$$3y - x = 6$$

The equations are equivalent.

Try This

One equation in a system is $y = 7x - 12$.

Complete the table to create systems with one solution, no solution, and infinite solutions.

	One Solution	No Solution	Infinite Solutions
System	$y = 7x - 12$ ________	$y = 7x - 12$ ________	$y = 7x - 12$ ________

Summary | Lesson 6

Systems of equations can represent *constraints* in a situation. There are different ways to solve systems of equations to determine the values that satisfy these constraints.

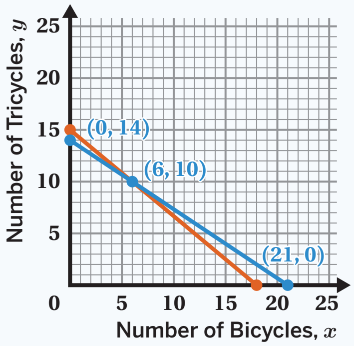
For example, a bike shop makes 2-wheel bicycles and 3-wheel tricycles. This week, they have 42 wheels and enough materials to make 16 bikes total.

Here is a system of equations about this situation:

- x is the number of bicycles
- y is the number of tricycles

$$x + y = 16$$
$$2x + 3y = 42$$

Graphing



The point of intersection (6, 10) is the solution.

Substitution

$$\begin{aligned} x + y &= 16 \\ -y \quad & \quad -y \\ x &= 16 - y \end{aligned}$$

$$\begin{aligned} 2x + 3y &= 42 \\ 2(16 - y) + 3y &= 42 \\ 32 - 2y + 3y &= 42 \\ \boxed{y = 10} \end{aligned}$$

$$x = 16 - 10$$
$$\boxed{x = 6}$$

Elimination

$$\begin{aligned} -2 \cdot (x + y &= 16) \\ 2x + 3y &= 42 \end{aligned}$$

$$\begin{aligned} -2x - 2y &= -32 \\ +\ 2x + 3y &= 42 \\ \hline 0 + y &= 10 \end{aligned}$$

$$\boxed{y = 10}$$

$$x + (10) = 16$$
$$\boxed{x = 6}$$

In this situation, the solution $x = 6$ and $y = 10$ means that the bike shop will use all of their wheels and materials if they make 6 bicycles and 10 tricycles.

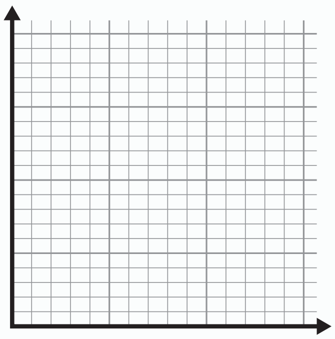
Try This

A new 50-unit apartment building has space for 80 parking spaces. The city requires 2 parking spaces for every multi-bedroom apartment, and 1 parking space for every one-bedroom apartment.

This situation can be modeled by this system of equations, where x represents the number of multi-bedroom apartments and y represents the number of one-bedroom apartments.

$$x + y = 50$$
$$2x + y = 80$$

a Graph the system of equations. Include a scale for the x and y axes.



b What does the solution represent in this situation?

Summary | Lesson 7

You can write systems of equations to help represent constraints in real-world situations.

Here are some things to consider when you write a system of equations:

- Identify the constraints of the situation.
- Identify what each variable will represent.
- Determine whether you want to use *standard form* ($Ax + By = C$) or *slope-intercept form* ($y = mx + b$) to represent each constraint.

Let's look at an example. The architect of an apartment building has enough space to fit 50 apartments and 80 parking spaces. The city requires 2 parking spaces for every big apartment, and 1 parking space for every small apartment.

Constraints	Variables	System of Equations
• There is space to fit 50 apartments and 80 parking spaces. • Every big apartment has 2 parking spaces and every small apartment has 1 parking space.	x represents the number of big apartments. y represents the number of small apartments.	$x + y = 50$ $2x + y = 80$

Try This

The knitting club sold 40 scarves and hats at a winter festival and made $\$700$. Each scarf costs $\$18$ and each hat costs $\$14$.

a If s represents the number of scarves sold and h represents the number of hats sold, which system of equations represents the constraints in this situation?

A. $40s + h = 700$
$18s + 14h = 700$

B. $18s + 14h = 40$
$s + h = 700$

C. $s + h = 40$
$18s + 14h = 700$

D. $40(s + h) = 700$
$18s = 14h$

b Solve the system of equations you chose.

c Describe what the solution means in this situation.

Summary | Lesson 8

You can solve systems of equations symbolically using either substitution or elimination. Looking for specific structures in the equations can help you decide which strategy to use.

- It may be helpful to use substitution when at least one of the equations has an isolated variable or at least one is in *slope-intercept form*.
- It may be helpful to use elimination when both equations are in the same form or if the equations have a pair of same or opposite terms.

When solving a system of equations symbolically, sometimes all of the variables are eliminated.

No Solutions

When the result is a false statement there are *no solutions* to the system of equations. This means the lines are parallel and will never intersect.

$$y = 3x + 6 \qquad y = 3x - 6$$

$$\begin{aligned} 3x + 6 &= 3x - 6 \\ 3x + 12 &= 3x \\ 12 &= 0 \end{aligned}$$

Infinitely Many Solutions

When the result is a true statement there are *infinitely many solutions* to the system of equations. The equations are equivalent and represent the same line.

$$\begin{aligned} 2 \cdot (2x + 4y &= 6) \\ -4x - 8y &= -12 \end{aligned}$$

$$\begin{array}{r} 4x + 8y = 12 \\ + \; -4x - 8y = -12 \\ \hline 0 + 0 = 0 \\ 0 = 0 \end{array}$$

Try This

Here are three systems of equations.

a Circle the strategy you would use to solve each system of equations.

$y = 3x - 10$ $2x - 3y = 16$	$y = 4x - 10$ $y = x + 2$	$9x + y = 25$ $3x + 2y = 5$
Elimination	Elimination	Elimination
Substitution	Substitution	Substitution
Both	Both	Both
Neither	Neither	Neither

b Select one system and solve it using the strategy you chose.

Summary | Lesson 9

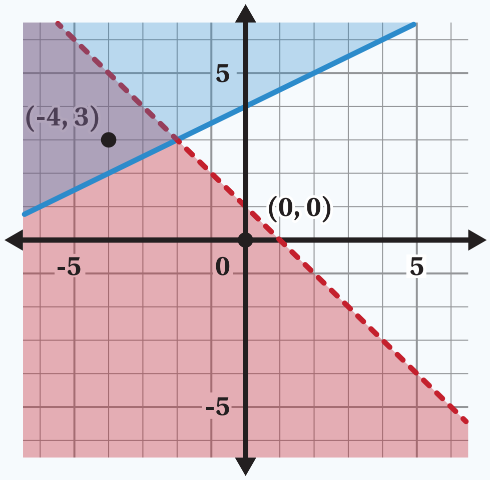
A **system of inequalities** is a system of two or more inequalities that represent the *constraints* on a shared set of variables.

You can use different strategies to determine if a point is a solution to a system of inequalities.

- If the point is in the shaded region for both inequalities, then it is a solution to the system.
- If the x- and y-values of the point are substituted into both inequalities and the inequalities are true, then the point is a solution to the system.

Here is a graph for this system of inequalities.

$$x + y < 1$$
$$y \geq \frac{1}{2}x + 4$$



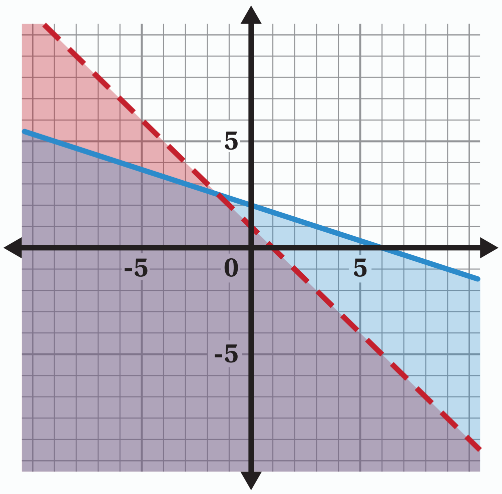
You can see that the point $(-4, 3)$ is a solution because it is in the shaded region for both inequalities.

You can also substitute points into both inequalities to determine if they are solutions. $(0, 0)$ is not a solution and $(-4, 3)$ is a solution.

Try This

This graph represents this system of inequalities:

$$x + 3y < 6$$
$$x + y < 1$$



a Is the point $(1, 0)$ a solution to the system?

b Select a point that does *not* represent a solution and show that it is not a solution.

Summary | Lesson 10

The **solutions to a system of inequalities** are all the points that make both inequalities true. The solutions can be seen in the region where the graphs overlap, called the **solution region**.

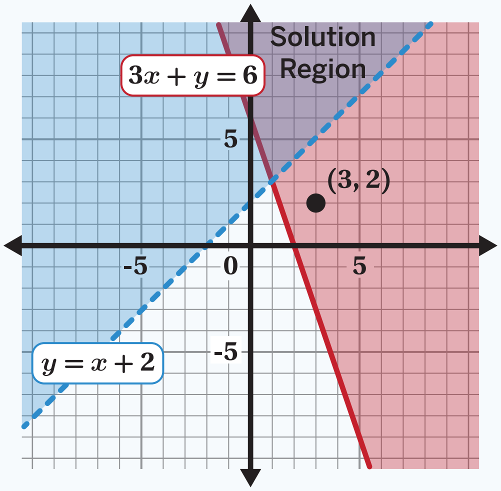
One strategy for determining the location of the solution region is to test a point. Choose a point that is not on either *boundary line*, substitute the x- and y-values into each inequality to see if it makes the statement true, and shade based on the results of the test.

Let's look an example for this system of inequalities:

$$3x + y \geq 6$$
$$y > x + 2$$

You can test the point (3, 2) to help determine the solution region.

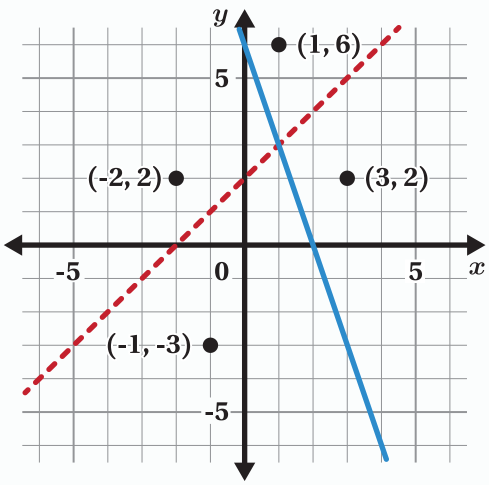
Solid Line	Dashed Line
$3x + y \geq 6$	$y > x + 2$
$3(3) + 2 \geq 6$	$0 > 3 + 2$
$11 \geq 6$	$0 > 5$
True ✓	False ✗
Shade the side of the solid line that includes $(3, 2)$	Shade the side of the dashed line that does not include $(3, 2)$



Try This

Here is a graph of this system of inequalities:

$$3x + y \geq 6$$
$$y > x + 2$$



a Determine if each point is a solution to the system. Circle yes or no.

$(-2, 2)$	Yes	No
$(1, 6)$	Yes	No
$(3, 2)$	Yes	No
$(-1, -3)$	Yes	No

b Shade the solution region on the graph.

Summary | Lesson 11

You can graph a system of linear inequalities by graphing the boundary line of each inequality and testing a point to determine which side of the boundary lines to shade.

You can use different strategies to help you graph boundary lines of inequalities.

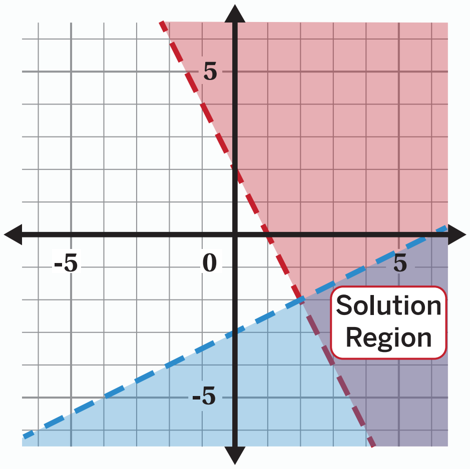
- A strategy to graph boundary lines written in *slope-intercept form* ($y = mx + b$) is to plot the y-intercept and use the slope to determine other points.
- A strategy to graph boundary lines written in *standard form* ($Ax + By = C$) is to plot and connect the x- and y-*intercepts*.

If an inequality uses a $\leq$ or $\geq$ symbol, then the boundary line is solid and included in the solution region. If an inequality uses a $<$ or $>$ symbol, then the boundary line is dashed and not part of the solution region.

You can also write a system of linear inequalities from a graph. Use the boundary lines and test points to help you determine the inequality symbols. You can also use test points to check the accuracy of your system of inequalities.

Try This

Here is a graph of a system of inequalities.



a Write the system of inequalities that this graph could represent.

b Explain how you decided which inequality symbols to use.

Summary | Lesson 12

Systems of inequalities can help you solve problems involving real-world *constraints*.

Here is an example about a juice shop.

A juice shop has 144 ounces of orange juice to fit in jars of 12 and 16 ounces. They earn \$2.50 for each 12-ounce jar and \$4.50 for each 16-ounce jar. They need to earn more than \$33.50 from the juice.

You can write and graph a system of inequalities to represent these constraints.

Let x represent the number of 12-ounce jars and y can represent the number of 16-ounce jars.

$$12x + 16y \leq 144$$

$$2.50x + 4.50y > 33.50$$

You can graph each boundary line and use the test point $(2, 7)$ to help us determine the solution region.

Solid Line

$$12x + 16y \leq 144$$
$$12(2) + 16(7) \leq 144$$
$$136 \leq 144$$

True ✓

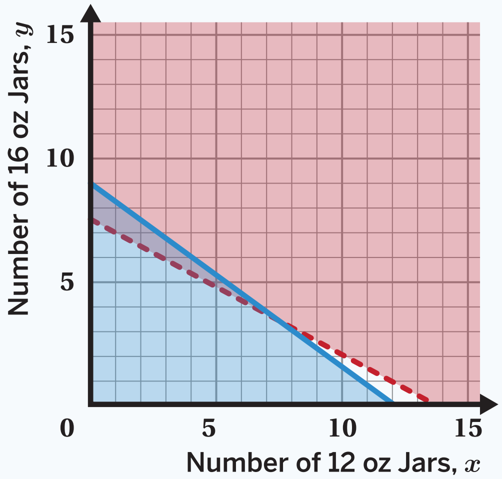
Shade the side of the solid line that includes $(2, 7)$

Dashed Line

$$2.50x + 4.50y > 33.50$$
$$2.50(2) + 4.50(7) > 33.50$$
$$36.50 > 33.50$$

True ✓

Shade the side of the dashed line that includes $(2, 7)$



You can use the graph to help you determine some possible combinations of 12- and 16-ounce jars of juice that meet the constraints. Such as:

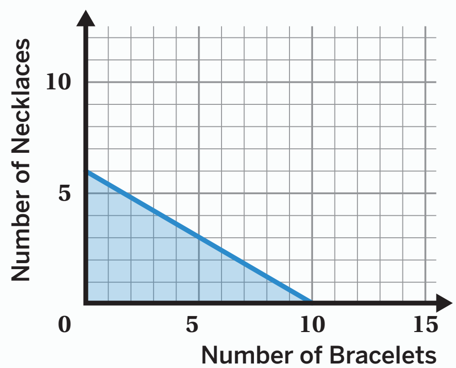
- Zero 12 oz jars of juice and eight 16 oz jars of juice
- Zero 12 oz jars of juice and nine 16 oz jars of juice
- One 12 oz jars of juice and eight 16 oz jars of juice

Try This

Natalia wants to buy bracelets and necklaces as gifts for her friends. Each bracelet, b, costs \$3 and each necklace, n, costs \$5. She can spend no more than \$30 and needs at least 7 gifts.

This graph shows one of the inequalities that represents the constraints in this situation:

$$3b + 5n \leq 30$$



a Write the second inequality that represents the constraints in this situation.

b Graph the inequality you wrote.

c Use the graph to determine two possible combinations of bracelets and necklaces that meet the constraints.

Lesson 1

a House $= 17$, Flower $=$ -7 , Drop $= 10$

b *Responses vary.*

- Calculate the value of the drops first because the first column is only drops and has a total of 30. If 3 drops is 30, then one drop is 10.
- Substitute the value of the drop, 10, to find the value of the house in the first row.
- Substitute the value of the drop, 10, and the value of the house, 17, to find the value of the flower.

Lesson 2

a Subtraction

b $p = 2, q = 4$

Caregiver Note:

$$\begin{array}{r} p + 3q = 14 \\ \text{-}(p + 2q = 10) \\ \hline q = 4 \end{array}$$

Subtract the second equation from the first equation to eliminate p.

$$\begin{array}{r} p + 2(4) = 10 \\ p + 8 = 10 \\ -8 \quad -8 \\ p = 2 \end{array}$$

After finding the value of q, substitute that value into the equation $p + 2q = 10$ to solve for p.

Lesson 3

$x = 0, y = 1$

Caregiver Note:

Strategy 1

$4x + 3y = 3 \Rightarrow \text{-}2(4x + 3y = 3)$

$8x + y = 1$

$$\begin{array}{r} \text{-}8x - 6y = \text{-}6 \\ 8x + y = 1 \\ \hline \text{-}5y = \text{-}5 \\ y = 1 \end{array}$$

Multiply the first equation by -2 to eliminate x and solve for y.

$$\begin{array}{r} 8x + (1) = 1 \\ -1 \quad -1 \\ \hline 8x = 0 \\ x = 0 \end{array}$$

Substitute 1 for y to solve for x.

Strategy 2

$4x + 3y = 3$

$8x + y = 1 \Rightarrow \text{-}3(8x + y = 1)$

$$\begin{array}{r} 4x + 3y = 3 \\ \text{-}24x - 3y = \text{-}3 \\ \hline \text{-}20x \quad = 0 \\ x \quad = 0 \end{array}$$

Multiply the second equation by -3 to eliminate y and solve for x.

$$\begin{array}{r} 4(0) + y = 1 \\ y = 1 \end{array}$$

Substitute 0 for x to solve for y.

Lesson 4

a $a = 2, b = \text{-}2$

Caregiver Note:

$$\begin{aligned} 2a + (3a - 8) &= 2 \\ 5a - 8 &= 2 \\ +8 \ \ +8 & \\ 5a &= 10 \\ a &= 2 \end{aligned}$$

Substitute the expression $(3a - 8)$ for b into the equation $2a + b = 2$ and then solve for a.

$$\begin{aligned} b &= 3(2) - 8 \\ b &= 6 - 8 \\ b &= \text{-}2 \end{aligned}$$

Then substitute the value of 2 for a, and solve for b.

b *Responses vary.*

- Both strategies involve substituting the value of one variable to determine the value of the other.
- Both strategies lead to the creation of new equations.
- The elimination strategy involves adding or subtracting equations.
- The substitution strategy is useful when one variable is already isolated in one or both equations.

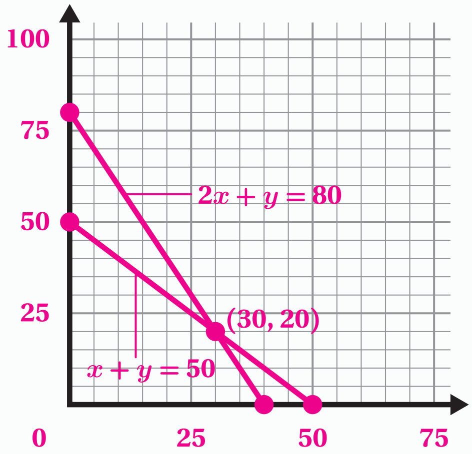
Lesson 5

Responses vary. Sample shown in table.

	One Solution	No Solution	Infinite Solutions
Equation	$y = 7x - 12$ $y = x - 4$ Any equation in $y = mx + b$ form where the coefficient of x is not 7 is correct.	$y = 7x - 12$ $y = 7x - 1$ Any equation in $y = mx + b$ form where the coefficient of x is 7 and the constant is not -12 is correct.	$y = 7x - 12$ $2y = 14x - 24$ Any equation equivalent to $y = 7x - 12$ is correct.

Lesson 6

a



b (30, 20) represents the city building 30 multi-bedroom apartments and 20 one-bedroom apartments.

Lesson 7

a C. $s + h = 40$
$18s + 14h = 700$

b $s = 35$, $h = 5$

Caregiver Note:

$s + h = 40 \Rightarrow -14(s + h = 40)$	$-14s - 14h = -560$
$18s + 14h = 700$	$18s + 14h = 700$
Multiply the first equation by -14 to eliminate the h and solve for s.	$4s = 140$ $s = 35$
Substitute 35 for s and solve for h.	$(35) + h = 40$ $-35 \quad -35$ $h = 5$

c The knitting club sold 35 scarves and 5 hats.

Lesson 8

a

$y = 3x - 10$ $2x - 3y = 16$	$y = 4x - 10$ $y = x + 2$	$9x + y = 25$ $3x + 2y = 5$
Elimination	Elimination	Elimination
Substitution	Substitution	Substitution
Both	Both	Both
Neither	Neither	Neither

b

$y = 3x - 10$ $2x - 3y = 16$	$y = 4x - 10$ $y = x + 2$	$9x + y = 25$ $3x + 2y = 5$
$(2, \text{-}4)$	$(4, 6)$	$(3, \text{-}2)$

Lesson 9

a No

Caregiver Note: The point $(1, 0)$ is on a dashed boundary line, which means it is not included in the solution region.

b *Responses vary.*

$(\text{-}5, 5)$

Caregiver Note:

$(\text{-}5) + 3(5) < 6$

$\text{-}5 + 15 < 6$

$10 < 6$ This is not true, so $(\text{-}5, 5)$ is not a solution.

Lesson 10

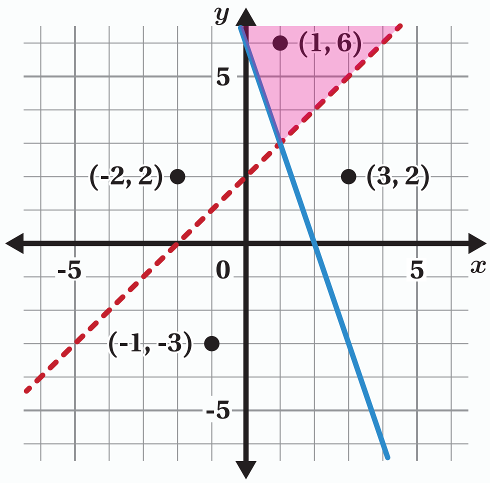
a (-2, 2): No

(1, 6): Yes

(3, 2): No

(-1, -3): No

b



Lesson 11

a $y > -2x + 2$ (or equivalent)

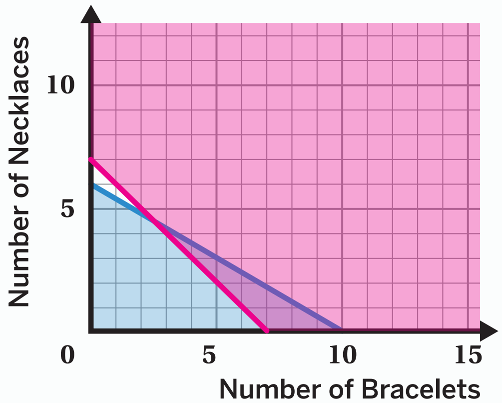
$y < \frac{1}{2}x - 3$ (or equivalent)

b *Responses vary*. The boundary lines are dashed, which means that the inequality symbols can only be less than (<) or greater than (>).

Lesson 12

a $b + n \geq 7$

b



c *Responses vary*. 6 bracelets and 2 necklaces, 7 bracelets and 1 necklace.