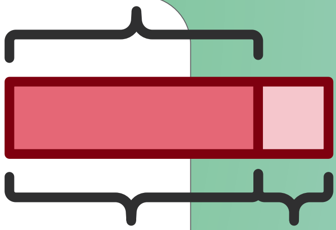


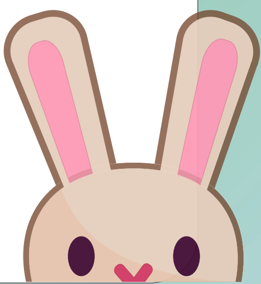
Unit 4

Proportional Relationships and Percentages

In this unit, you'll learn what happens when a quantity changes by a percentage. You will also use tape diagrams, tables, double number lines, and equations to solve proportional relationships involving percent change and fractional quantities.

Essential Questions

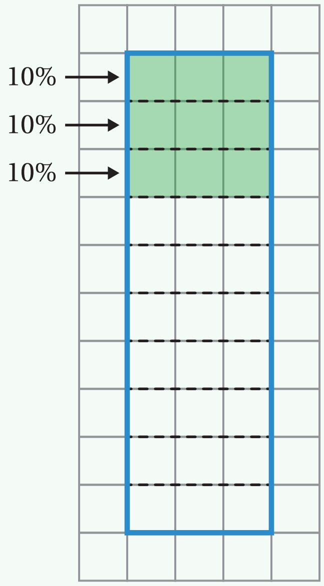
- How are percentages and fractions related to proportional relationships?
- How do proportional relationships and percentages represent change in the real world?



Summary | Lesson 1

When determining the percent of a region that's shaded:

- Create a fraction by counting the number of shaded units (part) as the numerator and the total number of units (whole) as the denominator.
- Divide the part by the whole and then change to a percent by multiplying by 100.



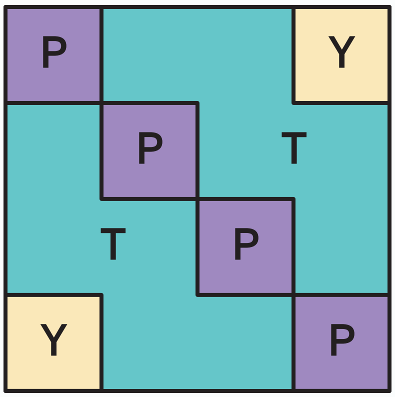
When calculating a percent of a number:

- Multiply the percent, written as a decimal or fraction, by the whole, or
- Partition a grid into equal groups to help you determine the part. For example, when determining 30% of 30, break 30 into 10 groups of 3, each representing 10%. For 30%, you would need 3 groups of 3, or 9.

Try This

What percentage of this square is shaded in each color?

Color	Percentage (%)
Purple P	
Teal T	
Yellow Y	



Summary | Lesson 2

Tape diagrams and tables can help us make sense of problems involving **percent increase** and **percent decrease**.

The terms *percent increase* and *percent decrease* describe an increase or decrease of a quantity as a percentage of the starting amount.

One method to solve these types of problems is to start with the original amount and then add or subtract the amount that matches the percent of increase or decrease.

Try This

Habib runs a snack stand in the park. Last week, he sold 80 water bottles.

The number of water bottles he sold increased by 75% this week.

How many water bottles did he sell this week?

Make a table or tape diagram if it helps with your thinking.

Summary | Lesson 3

We can use equations to help us make sense of situations involving *percent increase* or *percent decrease*.

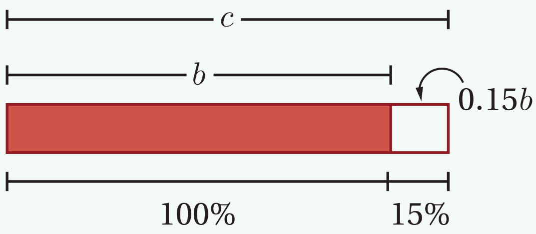
For example, c is 15% more than b.

Three equations can be written to model the relationship between b and c:

$c = b + 0.15b$

$c = (1 + 0.15)b$

$c = 1.15b$



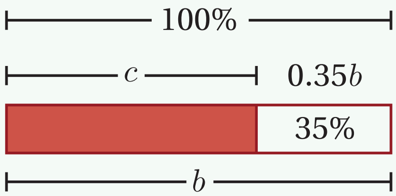
In this example, c is 35% less than b.

Three equations can be written to model the relationship between b and c:

$c = b - 0.35b$

$c = (1 - 0.35)b$

$c = 0.65b$



Try This

Martina’s available computer storage decreased by 3% this week.

Write an equation to represent the amount of storage space she had last week, b, and the amount she has this week, c.

Summary | Lesson 4

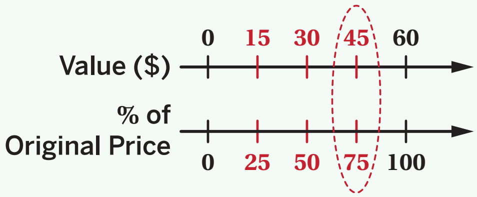
A double number line diagram is a helpful tool for understanding percentage problems.

When using double number lines, it helps to first identify which value aligns with which percentage. It may be helpful to think of the values as a new amount and an original, or old, amount.

Once the known values and percentages are aligned, filling in more values on both number lines can help you solve the problem. For percentage problems, the 0 on each number line should be aligned.

Example:

A furniture store offers 25% off every piece of furniture to make room in the warehouse. If a chair normally sells for \$60, what is its sale price?



Try This

A sports drink contains 40% less sugar than before. The drink now contains 18 grams of sugar.

a How much sugar did the drink originally contain?

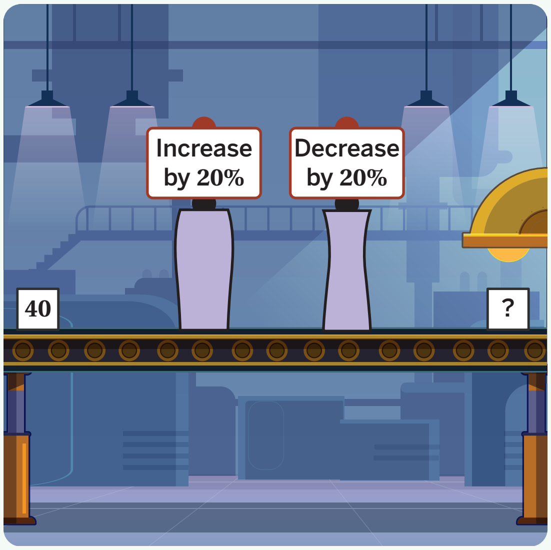
b Use a double number line to represent this situation.

Percent machines take an input value and increase or decrease that value by a percentage to produce an output.

Increasing and then decreasing by the same percentage will not produce the original input value.

For example:

- An input value of 40 is increased by 20%.
  $40 \cdot 1.2 = 48$ or $40 \cdot 0.2 + 40 = 48$
- The new value is decreased by 20%.
  $48 \cdot 0.8 = 38.4$ or $48 - (48 \cdot 0.2) = 38.4$



It may surprise you to discover the final result is *not* the original input value of 40! This is because the input value represents the whole when you determine 20% of 40. But in the second calculation, the input value changes. You are now determining 20% of 48 and then subtracting that value from 48.

Try This

A number is increased by 30%. The new number is 26.

What was the original number?

Use a tape diagram, double number line, or table to show your thinking.

Summary | Lesson 6

Prices generally go up over time, but *how much* prices go up matters. Dollar amounts and percent changes are two ways of describing price increases. Percentages are more useful when comparing price changes of two or more different things. Let's examine the price of bananas and ground beef.

One way to calculate the percent increase in price for each item is to divide the new price by the old price.

- Bananas: $\frac{0.63}{0.50} = 1.26$. This means 126%, or a 26% increase.
- Ground beef: $\frac{5.21}{2.56} = 2.04$. This means 204%, or a 104% increase.

Year	Pound of Bananas	Pound of Ground Beef
2004	$0.50	$2.56
2024	$0.63	$5.21

The price for both items increased during this period, but the price of ground beef increased much more dramatically than the price of bananas.

It can be useful to see how prices change over time compared to people's income. The federal minimum wage is an example of a measure of income. From 2004 to 2024, minimum wage changed from $5.15 to $7.25, which is a 41% increase. That means that for someone making minimum wage in 2004 and 2024, bananas would feel like they have gotten cheaper with time, but ground beef would feel more expensive.

Try This

Here is a table of Annika's wages and rent for two years.

Year	Wage ($ per hr)	Monthly Rent ($)
2012	12.50	550
2022	18.75	

a By what percent did Annika's wages increase from 2012 to 2022?

b The cost of Annika's rent increased by 220% from 2012 to 2022. Complete the table.

Summary | Lesson 7

The listed price and the total a customer ends up paying are often two different quantities. Tax, tip, and discounts are some of the reasons why the listed price and final price are different.

These changes are often calculated as percentages of the listed price.

For example, a tip is an amount of money that a person gives someone who provides a service, such as restaurant servers, hairdressers, and delivery drivers. If a person plans to leave a 20% tip, then the total cost with tip will be 120% of the bill.

Try This

Imani has a coupon for 25% off a hat.

a Determine the price of the hat before tax.

b Sales tax in Imani's state is 6%.

What was the total amount Imani spent including tax?

Price:	$14.00
25% Off Coupon:	$
Subtotal:	$
6% Tax:	$
Total:	$

Summary | Lesson 8

There are two different federal minimum wages — one for workers who receive tips and one for workers who don't.

For tipped workers, such as restaurant servers, pay depends not only on their hourly wage and the number of hours they work, but also the number of tables served, the average bill at those tables, and the average percent tip.

Let's say a restaurant server earns $2.13 per hour and works 30 hours per week. They serve about 40 tables in a week, where the typical bill is $75 and the tip is 18%. Here's an equation that shows how much this server earns in a typical week:

$$2.13 \bullet 30 + 40 \bullet 75 \bullet 0.18 = \$603.90$$

We can compare this amount to other ways of paying servers, such as a simple rate of $15 per hour ($15 \bullet 30 = \$450$ in a week) with no tips. Switching to this way would mean about a 25% decrease in pay for the restaurant server in our example because $\frac{603.90 - 450}{603.90} \approx 0.2548$.

Try This

Demetrius is a server at a restaurant. In an average 40-hour work week, he serves 60 tables, with an average bill of $28 per table. He typically receives a 20% tip on each bill and earns $2.13 per hour.

a How much money does Demetrius earn in a typical week?

b Let's say the typical tip decreased to 18% of the bill.
By what percent would Demetrius' earnings decrease?
Show or explain your thinking.

Percent error describes the difference between a desired value and the actual value, expressed as a percent of the desired value.

For example, a milk carton is supposed to contain 16 fluid ounces, but it only contains 15 fluid ounces.

- The error is 1 fluid ounce.
- The percent error is 6.25% because $\frac{1}{16} \cdot 100 = 6.25$.

To determine the percent error, the amount of the error is compared to the desired value. You can use this formula:

$$\text{Percent error} = \frac{\text{(difference between actual value and the desired value)}}{\text{desired value}} \cdot 100$$

In some situations, there is no clear "desired" value. In those cases, the denominator is the value that has no error. Here are some examples:

- For a thermometer that reads $73°$, if the real temperature is $70°$, the percent error for that thermometer's reading is $\frac{3}{70}$, or 4.3%.
- For an estimate of 800 jelly beans in a jar, if the jar actually has 947 jelly beans, the percent error of the estimate is $\frac{147}{947}$, or 15.5%.

Try This

Anushka makes bracelets to sell at a children's shop. A child-size bracelet should be 14 centimeters long. If a bracelet is more than 15% longer or shorter, it must be resized.

a What is the longest bracelet Anushka can make that will be considered child-size?

b If Anushka makes a bracelet that is 12 centimeters long, will she need to resize it?

Show or explain your thinking.

Summary | Lesson 10

Situations involving percent increase and decrease are everywhere in our society.

For example, news articles often contain facts and statistics about pollution, such as:

In 2019, the U.S. generated 72.8 million tons of plastic waste.
This was 55% more waste than in 2000.

Information like this can be used to generate interesting questions. When writing these questions, it's important to be precise with language. We could ask: *How much waste was there in 2000?* But a more precise question might be: *How many tons of plastic waste did the U.S. generate in 2000?*

We can use strategies from this unit, such as equations, double number lines, tables, and tape diagrams, to answer these kinds of questions.

Try This

Here is some information about pollution from the summary.

In 2019, the U.S. generated 72.8 million tons of plastic waste.
This was 55% more waste than in 2000.

a How many tons of plastic waste did the U.S. generate in 2000?

b Pose another question about this topic that you are interested in answering.

Summary | Lesson 11

When a problem involves a proportional relationship, determining the constant of proportionality or scale factor can be helpful.

You can see this relationship between the columns of this table. The heights are multiplied by $2\frac{1}{2}$ to get the widths.

Height (in.)	Width (in.)
2	5
$3\frac{1}{2}$	$8\frac{3}{4}$

$\cdot 2\frac{1}{2}$

$\cdot 2\frac{1}{2}$

When you multiply one quantity in a proportional relationship by a value, the other quantity will change by the same factor whether or not the values are whole numbers.

You can see this relationship between the rows of the table. When the height is multiplied by $1\frac{3}{4}$, the width is multiplied by the same number.

Height (in.)	Width (in.)
2	5
$3\frac{1}{2}$	$8\frac{3}{4}$

$\cdot 1\frac{3}{4}$

$\cdot 1\frac{3}{4}$

Try This

Hamza wants to create wearable pins that are a scaled copy of his original design.

The pins will be $\frac{4}{5}$ inches tall. Complete the table to show the width of his pins.

	Height (in.)	Width (in.)
Design	2	5
Pin		

Proportional relationships may involve fractional amounts. You can solve problems involving fractions by using the same strategies you use to solve problems with whole numbers.

- To determine the constant of proportionality within a recipe, divide the amount of an ingredient by the total number of servings.
- Constants of proportionality can help to compare proportional relationships involving fractional quantities.

Here is a recipe for banana bread. To find the amount of sugar per serving, divide $\frac{3}{4}$ cups of sugar by 6 servings. This gives you $\frac{3}{4} \div 6$, or $\frac{1}{8}$ cups of sugar per serving.

Banana Bread Recipe

Number of servings: 6

- 2 lb of bananas
- $\frac{1}{2}$ cups of butter
- $\frac{3}{4}$ cups of sugar
- $2\frac{1}{2}$ cups of flour
- 1 tsp of baking soda

Try This

Kwasi is making banana bread.

He wants to make a larger loaf to serve 10 people. How much of each ingredient will he need?

Show or explain your thinking.

Kwasi's Recipe

Number of servings: 6

- 2 lb of bananas
- $\frac{1}{2}$ cup of butter
- $\frac{3}{4}$ cup of sugar
- $2\frac{1}{2}$ cups of flour
- 1 tsp of baking soda

Long division can be used to represent fractions as decimals. Sometimes, a decimal is a **terminating decimal**, which means that it ends. Other times, a decimal is a **repeating decimal**, where one or more of its digits (not all zeros) repeats forever. A repeating decimal can be written using bar notation over the digits that repeat or with the ellipses (...) at the end.

Examples

$\frac{1}{3} = 0.333\ldots = 0.\overline{3}$

$\frac{14}{99} = 0.14141414\ldots = 0.\overline{14}$

$\frac{53}{90} = 0.5888\ldots = 0.5\overline{8}$

Remember to put the bar *only* over the repeating digit(s).

Try This

Use long division to write each number as a decimal. Then determine if each decimal is terminating or repeating. Explain how you know.

a $\frac{7}{8}$

b $\frac{2}{3}$

Try This | Answer Key

Lesson 1

Color	Percentage (%)
Purple P	25
Teal T	62.5
Yellow Y	12.5

Caregiver Note: Here is a strategy to determine the percent that is shaded in purple: Since 4 squares are purple and there are 16 squares in total, write the fraction $\frac{4}{16}$ and divide $4 \div 16 = 0.25$. Multiply by 100 to get 25%.

Lesson 2

140 water bottles.

Caregiver Note: One strategy is to start with the original amount, 80 water bottles, then calculate 75% of it. 75% of 80 is 60 ($0.75 \cdot 80 = 60$). Since this is an increase, add that amount to the original. $80 + 60 = 140$ water bottles.

Lesson 3

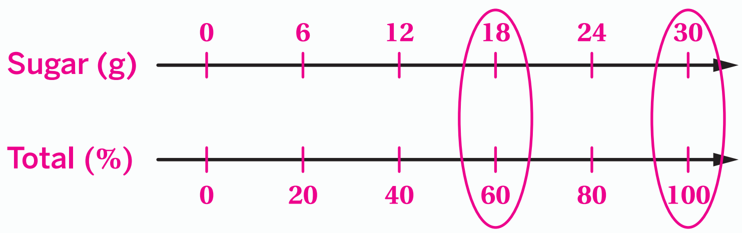
Responses vary.

- $c = b - 0.03b$
- $c = (1 - 0.03)b$
- $c = 0.97b$

Lesson 4

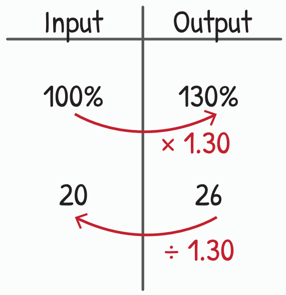
a 30 grams of sugar

b



Lesson 5

20. *Work varies.* Caregiver Note: One strategy to determine the original number is to create a table and use the constant of proportionality.



Lesson 6

a 50%.
Caregiver Note: Here is one way to calculate the percent increase in wages: $\frac{18.75}{12.50} = 1.50$*. This means* 150%*, or a* 50% *increase.*

b

Year	Wage ($ per hr)	Monthly Rent ($)
2012	12.50	550
2022	18.75	1,210

Caregiver Note: Here is one way to find the missing number: $220\% = 2.20$. *To calculate* 220% *of* 550, *multiply* $550 \cdot 2.2 = 1210$.

Lesson 7

a $10.50

b $11.13

Lesson 8

a $421.20. $(40 \cdot 2.13) + (60 \cdot 28 \cdot 0.2) = \$421.20$

b 8%. *Explanations vary.* $(40 \cdot 2.13) + (60 \cdot 28 \cdot 0.18) = \$387.60$. $\frac{387.60}{421.20} \approx 0.92$, so the new earnings are 92% of the original. That's an 8% decrease.

Lesson 9

a 16.1 centimeters.

Caregiver Note: One strategy for finding a maximum is to multiply the desired length by the percent error plus 100%. $14 \cdot 1.15 = 16.1$, so that is the longest a bracelet can be without resizing.

b No. *Responses vary.* Bracelets can be up to 15% shorter, or $100\% - 15\% = 85\%$ of the desired length. 85% of 14 is $14 \cdot 0.85 = 11.9$. 12 centimeters is larger than 11.9 centimeters, which means the bracelet is within the acceptable range.

Lesson 10

a About 47 million tons. 72.8 million tons is 55% more than the amount in 2000.

Caregiver Note: One strategy to find the original amount is to divide $\frac{72.8}{1.55} \approx 47$.

b *Responses vary.*

- How many tons of plastic waste did the U.S. generate last year?
- By what percent did the amount of plastic waste change from the year 2000 to last year?

Lesson 11

	Height (in.)	Width (in.)
Design	2	5
Pin	$\frac{4}{5}$	2

$2 \xrightarrow{\times \frac{5}{2}} 5$

$\frac{4}{5} \xrightarrow{\times \frac{5}{2}} 2$

Lesson 12

$3\frac{1}{3}$ pounds of bananas

$\frac{5}{6}$ cups of butter

$1\frac{1}{4}$ cups of sugar

$4\frac{1}{6}$ cups of flour

$1\frac{2}{3}$ tsp of baking soda

Explanations vary. One strategy is to multiply each original amount by $\frac{10}{6}$.

For instance, to determine the new amount of butter, multiply $2 \cdot \frac{10}{6} = \frac{20}{6} \div \frac{2}{2} = \frac{10}{3} = 3\frac{1}{3}$ cups.

Lesson 13

a

$$\begin{array}{r} 0.875 \\ 8\overline{)7.000} \\ \underline{64} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Terminating. *Explanations vary*. This decimal stops and does not repeat.

b

$$\begin{array}{r} 0.666 \\ 3\overline{)2.000} \\ \underline{18} \\ 20 \\ \underline{18} \\ 20 \\ \underline{18} \\ 2 \end{array}$$

Repeating. *Explanations vary*. This decimal will go on forever in a pattern of sixes.